NOTE ON THE EXISTENCE OF THE VON NEUMANN GROWTH EQUILIBRIUM AND THE KUHN-TUCKER THEOREM

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1. There is a huge literature concerning the existence of the von Neumann growth equilibrium. Here is presented another proof, which may be of some use in a class of mathematical economics since it avoids the fixed–point theorems and makes a direct use of the Kuhn–Tucker theorem (K–H theorem hereinafter) in nonlinear programming [5]. Such a proof is implicitly suggested in Karlin [3], Lancaster [6] and Morishima [7] where the existence problem is formulated in terms of nonlinear programming. They, however, prove the existence by using variations of the Tucker theorem in linear systems or the separation theorem. Now it is well known that optimal progamming, the separation theorems and the fixed–point theorems, all these are closely related to one another in various economic problems.

Though the K–H theorem, based on the Minkowski–Farkas' lemma which in turn is derived from the Tucker theorem, is rather sophisticated for average students, the author thinks it desirable to teach the K–H theorem in the early stage of lecture course and then, utilizing this thorem, to prove the Frobenius theorem for non–negative matrices (see [8]), the existence of the von Neumann equilibrium and so on.

By adopting nonlinear programming approach, we can enjoy one more merit. Namely, it allows us a direct interpretation of the maximum growth rate and the minimum interest rate in a balanced growth.

One remark is in order. In this note, we are not concerned with the uniqueness of the equilibrium. This topic is dealt with in [2].

2. The von Neumann equilibrium is described as follows:

(1) $Bx \geqq \alpha Ax$,

(2) $pB \leqq \beta pA$,

(3) $pBx = \alpha pAx$, (the rule of free goods),

(4) $pBx = \beta pAx$, (the rule of profitability).

Here, B is an output matrix, while A is an input matrix. x shows a column vector of relative activity levels and p a row vector of relative prices. All matrices and vectors are naturally non–negative. The meaning of the above equations are to be found in [9], [1], [7] etc.

We assume that each row of B has at least one positive element and each column of A has at least one positive element. The implication of these assumptions should be clear [4].

3. Now we have to examine whether meaningful vectors, $x \geq 0$ and $p \geq 0$, exist for the above system(1)—(4). To do so, let us formulate the following problem.

Problem: maximize α subject to equation (1) and

(5) $ex = 1$.

where α and x are variables and $e = (1, 1, \cdots, 1)$. First form the Lagrangian function:

$$L = \alpha - \mu(\alpha Ax - Bx) - \lambda(ex - 1).$$

μ is a row vector and λ a scalar multiplier. By the assumption, there is at least one solution pair, $\alpha^* > 0$ and $x^* \geq 0$, which may or may not be unique.

And we can apply the K–H theorem. At a solution point, there can be found μ^* and λ^* such that

$$(6)\quad 1-\mu^*Ax^*=0,$$

$$(7)\quad -\mu^*\alpha^*A+\mu^*B-\lambda^*e\leqq 0,$$

$$(8)\quad -(\alpha^*Ax^*-Bx^*)\geqq 0,$$

$$(9)\quad (-\mu^*\alpha^*A+\mu^*B-\lambda^*e)x^*=0,$$

$$(10)\quad -\mu^*(\alpha^*Ax^*-Bx^*)=0.$$

From (9) and (10), we know $\lambda^*=0$ since $ex^*=1$. This implies the constraint (5) is inessential. Now we can rewrite (7)—(10) as follows:

$$\mu^*B\leqq\alpha^*\mu^*A,$$

$$Bx^*\geqq\alpha^*Ax^*,$$

$$\mu^*Bx^*=\alpha^*\mu^*Ax^*.$$

Thus, by putting $\alpha=\beta=\alpha^*$, $x=x^*$, $p=\mu^*$ ($\geqq 0$ by (6)) in the system (1)—(4), we obtain an equilibrium set of variables. Equations (3), (6) tell that the value of output is positive.

Alternatively we could formulate the minimization program:

Problem: minimize β subject to equation (2) and

$$pe'=1.$$

In the way similar to the above, we can find an equilibrium where the minimum interest rate is observed.

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